

Remarks on d-gonal curves

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REMARKS ON d -GONAL CURVES

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§ 0. Introduction.

Let M be a compact Riemann surface and f be a meromorphic function on M . We denote the principal divisor associated to f by (f) and the polar divisor of f by $(f)_\infty$. If $d = \text{degree of the divisor } (f)_\infty$, we call f a meromorphic function of degree d . If d is the minimal integer in which a non-trivial meromorphic function f of degree d exists on M , then we call M a d -gonal curve. In this case the complete linear system $| (f)_\infty |$ has projective dimension one. Moreover if f defines a cyclic covering $M \rightarrow P_1$ over a Riemann sphere P_1 , then we call M a cyclic d -gonal curve.

Now we assume that M is a p -gonal curve of genus g with a prime number p . Then Namba has shown that M has a unique linear system g_p^1 of projective dimension one and degree p provided $g > (p-1)^2$ ([6]). For example if M is defined by an equation $y^p - (x-a_1)^{r_1} \cdots (x-a_s)^{r_s} = 0$ with $(p, r_i) = 1$, $\sum r_i \equiv 0 \pmod{p}$ and $s \geq 2p+1$, then M is p -gonal and having a unique g_p^1 ([7]).

In this paper we treat a compact Riemann surface M defined by an equation ;

$$y^d - (x-a_1)^{r_1} \cdots (x-a_s)^{r_s} = 0 \quad *)$$

$$\text{with } \sum r_i \equiv 0 \pmod{d} \text{ and } 1 \leq r_i < d,$$

where d is not necessarily a prime number.

In § 2, we will show that M is d -gonal with the function x of degree d if there are enough r_i 's relatively prime to p for each prime number p dividing d . In this case we call M a cyclic d -gonal curve. We will also show that M has a unique g_d^1 if there are more sufficient such r_i 's as above (§ 2).

In § 3, let M be a cyclic d -gonal curve defined by $*)$ having a unique g_d^1 and M' be a compact Riemann surface defined by $y^d - (x-b_1)^{t_1} \cdots (x-b_s)^{t_s} = 0$. We will study the relations among a_i , b_i , r_i and t_i ($1 \leq i \leq s$) in the case M and M' are conformally equivalent. Namba [7] and Kato [5] have already studied this problem in the case d is a prime number. We will give similar results for an arbitrary d (§ 3).

In §4, we consider a covering map $\pi': M' \rightarrow M$, where M is a cyclic d -gonal curve with a unique g_d^1 and M' is a d' -gonal curve. In the case $d=d'$, we can apply the same methods in [3], and we will see that M' is also cyclic d -gonal. Moreover if π' is normal and $d=d'$, then the covering group of π' is isomorphic to *cyclic, dihedral, tetrahedral, octahedral* or *icosahedral*. For a general case $d \leq d'$, we will show some relations between d and d' (§4).

In §5, we will give some remarks about coverings $M \rightarrow N$ with a cyclic d -gonal curve M having a unique g_d^1 .

Finally we determine the equation $*$), which defines the curve M (with a unique g_d^1) having an automorphism $V (\notin \langle T \rangle)$ of order N , where T is the automorphism defined by $T^*x=x$ and $T^*y=e^{2\pi i/d}y$ (§6).

§ 1. Preliminaries

At first we give several results on the existence of meromorphic functions on a compact Riemann surface M of genus g following Accola and Namba.

LEMMA 1.1. (Accola [1]) *Let M be a compact Riemann surface of genus g . Let f_1 and f_2 be two meromorphic functions on M of degree n_1 and n_2 respectively. If f_1 and f_2 generate the full field $C(M)$ of meromorphic functions on M , then $g \leq (n_1-1)(n_2-1)$.*

The following lemma by Namba is easily obtained from Lemma 1.1.

LEMMA 1.2 (Namba [6]) *Let M be a compact Riemann surface of genus g and f be a meromorphic function of degree p on M with a prime number p .*

(1) *If h is a meromorphic function of degree n on M satisfying $(p-1)(n-1) \leq g-1$, then p divides n and $h=r(f)$, where $r(x)$ is a rational function of degree n/p .*

(2) *If $(p-1)^2 \leq g-1$, then M is p -gonal and having a unique linear system g_p^1 of degree p and dimension 1.*

PROOF. (1) By lemma 1.1, the subfield $C(f, g)$ of $C(M)$ generated by f and h is not equal to $C(M)$. As $p=[C(M):C(f)]$ is a prime number, $C(f)=C(f, g)$. (2) If h is any meromorphic function of degree p , then $C(h)=C(f)$ by (1). \square

Next we give some results concerning covering maps. Let $\pi: M' \rightarrow M$ be an arbitrary covering with compact Riemann surfaces M and M' . For a divisor $D=\sum n_i Q_i$ ($n_i \in \mathbb{Z}$, $Q_i \in M'$) we define a divisor $Nm_x D = Nm D$ by $\sum n_i \pi(Q_i)$. On

the other hand, for a meromorphic function f on M' we denote by $Nm[f]$ the meromorphic function on M obtained by the norm map $Nm: C(M') \rightarrow C(M)$. It is well known that the equation of principal divisors $Nm_x(f) = (Nm[f])$ holds ([2]). When the divisor $Nm(f)$ is trivial, we can choose a constant c such that the divisor $Nm(f+c)$ is non trivial. This means that $d' \geq d$ if M' and M are d' -gonal and d -gonal respectively.

When M and M' are both d -gonal, we have the following lemma:

LEMMA 1.3 (Ishii [3]) *Let $\pi': M' \rightarrow M$ be a covering map that both M and M' are d -gonal. Then;*

(1) *there exists a covering map $\pi: P'_1 \rightarrow P_1$ with Riemann spheres P'_1 and P_1 satisfying the following diagram;*

$$\begin{array}{ccc} M' & \xrightarrow{\phi'} & P'_1 \\ \pi' \downarrow & & \downarrow \pi \\ M & \xrightarrow{Nm[\phi']} & P_1 \end{array}, \quad \begin{array}{l} C(M') = C(M) \otimes_{C(P_1)} C(P'_1), \\ C(M) \cap C(P'_1) = C(P_1), \end{array}$$

where ϕ' is a morphism of degree d ,

(2) *if M' has a unique g_d^1 and π' is normal, then π is also normal and $Gal(M'/M) \cong Gal(P'_1/P_1)$ (i.e., cyclic, dehdral, tetrahedral, octahedral, or isosahedral).*

§ 2.

Let M be a compact Riemann surface of genus g that has two meromorphic functions h and h' of degree d and d' respectively. Let $C(h, h')$ be a subfield of $C(M)$ generated by h and h' , and \tilde{M} be the compact Riemann surface of genus \tilde{g} whose function field is isomorphic to $C(h, h')$. Put $[C(M): C(h, h')] = t$. Then \tilde{M} has meromorphic functions of degree d/t and d'/t induced by h and h' respectively. By Lemma 1.1 we have;

LEMMA 2.1. $\tilde{g} \leq (d/t - 1)(d'/t - 1)$.

From now on we assume;

M is defined by the equation $*$), T is the automorphism of M defined by $(x, y) \mapsto (x, \zeta_d y)$, where $\zeta_d = \exp(2\pi i/d)$, and h is the canonical map $M \rightarrow M/\langle T \rangle = P_1$.

We denote by g_k the genus of the quotient compact Riemann surface $M/\langle T^k \rangle$ for a positive integer k dividing d and $k \neq d$. Moreover if $k = q$ is a prime

number, we denote by s_q the number of branch points of the canonical map $M/\langle T^q \rangle \rightarrow M/\langle T \rangle \cong P_1$. s_q is equal to the number of r_i 's prime to q and we have $g_q = (q-1)(s_q-2)/2$ ($\because \sum r_i \equiv 0 \pmod d$).

LEMMA 2.2. Assume that M has a meromorphic function h' of degree d' . Let q_0 be the smallest prime number dividing $G.C.D. (d, d') = (d, d')$. If d' satisfies the inequalities:

$$g_q > (d/q_0 - 1)(d'/q_0 - 1) \cdots \cdots **)$$

for any prime q dividing $G.C.D. (d, d')$,

then $t = d$ or 1. Especially when $(r_i, d) = 1$ for all $1 \leq i \leq s$, $t = d$ or 1 provided $g_{q_0} > (d/q_0 - 1)(d'/q_0 - 1)$.

PROOF. Assume $t \neq d, 1$. As $\langle T^{d/t} \rangle$ is a unique subgroup of order t in $\langle T \rangle$, \tilde{M} should be isomorphic to $M/\langle T^{d/t} \rangle$ and $\tilde{g} = g_{d/t}$. For any prime number q dividing d/t ($\neq 1$), we have $\langle T^q \rangle \supset \langle T^{d/t} \rangle$ and $\tilde{g} - 1 \geq g_q - 1 \geq (d/q_0 - 1)(d'/q_0 - 1) \geq (d/t - 1)(d'/t - 1)$. This contradicts to Lemma 2.1. If $(r_i, d) = 1$ for all $i = 1, \dots, s$, then $s = s_q = s_{q_0}$ and $g_q \geq g_{q_0}$ for any prime number q dividing (d, d') . Thus the latter part of this lemma is reduced to the first part. \square

PROPOSITION 2.3. Assume M is a compact Riemann surface of genus g defined by the equation $*$). Let d' be a positive integer satisfying the inequalities $**$) in lemma 2.2 and $(d-1)(d'-1) \leq g-1$. Then;

(1) If d does not divide d' , then there is no meromorphic function of degree d' .

(2) If d divides d' , then every meromorphic function h' of degree d' is obtained by $r(h)$, where r is some rational function of degree d'/d and h is the canonical map $M \rightarrow M/\langle T \rangle$.

PROOF. Let h' be a meromorphic function of degree d' . $(d-1)(d'-1) \leq g-1$ means $t \neq 1$ by lemma 1.1. Thus $C(h, h') = C(h)$ by lemma 2.2 and $h' = r(h)$ for some rational function r . \square

REMARK. If $d = p$ is a prime number, this proposition is exactly same as Lemma 1.2(1).

THEOREM 2.4. Let M be a compact Riemann surface of genus g defined by $*$) and q_0 be the smallest prime number dividing d .

(1) Assume $(d-1)(d-2) \leq g-1$ and $(d/q_0 - 1)(d/q_0 - 2) \leq g_q - 1$ for any prime q dividing d . Then M is d -gonal.

(2) Assume $(d-1)^2 \leq g-1$ and $(d/q_0-1)^2 \leq g_q-1$ for any prime q dividing d . Then M is d -gonal and having a unique g_d^1 .

PROOF. (1) Assume that there is a meromorphic function h' of degree d' with $d' \leq d-1$. By $(d-1)(d-2) \leq g-1$ and lemma 1.1, $t=[C(M):C(h, h)] \neq 1$. As $t|(d, d')$ and $d' < d$, we have $d' \leq d-t$. Thus $d'/q_0 \leq d/q_0-1$ and $(d/q_0-1)(d'/q_0-1) \leq (d/q_0-1)(d/q_0-2) \leq g_q-1$ for any prime number q dividing d . Hence the assumptions in Proposition 2.3 are satisfied. This is a contradiction. (2) Let h' be a meromorphic function of degree d . By the same way as in (1) and Proposition 2.3(2), we have $C(h, h')=C(h)$. Thus M has a unique g_d^1 . \square

When $(r_i, d)=1$ for all $i=1, \dots, s$, we can restate Theorem 2.4 as follows;

THEOREM 2.4'. (1) If $(d-1)(d-2) \leq g-1$ and $(d/q_0-1)(d/q_0-2) \leq g_{q_0}-1$, then M is d -gonal.

(2) If $(d-1)^2 \leq g-1$ and $(d/q_0-1)^2 \leq g_{q_0}-1$, then M is d -gonal and having a unique g_d^1 .

PROOF. Use the latter part of Lemma 2.2. \square

EXAMPLE 2.5. Let M be a compact Riemann surface defined by $y^4 - x(x-a_1)(x-a_2)(x-a_3)\{(x-a_4)(x-a_5)(x-a_6)(x-a_7)\}^2 = 0$, where a_i ($1 \leq i \leq 7$) are distinct non-zero numbers, then $g=7$. Put $N=M/\langle T^2 \rangle$. N is defined by $y^2 - x(x-a_1)(x-a_2)(x-a_3) = 0$, i.e., $g_2=1$. M satisfies the conditions of Theorem 2.4(1), and then M is 4-gonal. On the other hand M has infinitely many g_i^1 . In fact if g_2^1 and $g_2^{1'}$ are two distinct linear systems on N , then $\pi^*g_2^1$ and $\pi^*g_2^{1'}$ are distinct linear systems of degree 4 and dimension 1 on M , where $\pi: M \rightarrow N$ is a canonical map. Thus M has infinitely many g_i^1 .

EXAMPLE 2.6. For prime numbers p and q with $p \geq q$, let M be defined by $y^{pq} - (x-a_1)^{r_1}(x-a_2)^{r_2} \dots (x-a_s)^{r_s} = 0$ with $\sum r_i \equiv 0 \pmod{pq}$ and $(r_i, pq)=1$, $1 \leq i \leq s$. If s satisfies $s \geq 2pq-1$ and $(p-1)(p-2) < (q-1)(s-2)/2$, then M is pq -gonal. If s satisfies $s \geq 2pq+1$ and $(p-1)^2 < (q-1)(s-2)/2$, then M is pq -gonal and having a unique g_{pq}^1 .

PROOF. These results are easily from $g=(pq-1)(s-2)/2$, $g_p=(p-1)(s-2)/2$, $g_q=(q-1)(s-2)/2$, and Theorem 2.4'. \square

EXAMPLE 2.7. Let M be defined by $y^4 - x^2(x-a_1)(x-a_2)(x-a_3) = 0$, where a_1, a_2, a_3 are distinct non-zero numbers. The covering map $x: M \rightarrow P_1$ is

completely ramified at A_1, A_2, A_3 and Q with $x(A_i)=a_i$ ($i=1, 2, 3$) and $x(Q)=\infty$ respectively. Also x is ramified at two points P_1 and P_2 with ramification index 2 and $x(P_1)=x(P_2)=0$. Thus $g=4(<(4-1)(4-2))$ and $g_2=1$. Then this M does not satisfy the conditions in Theorem 2.4(1). In fact M is trigonal with a principal divisor $(x/y)=P_1+P_2+Q-A_1-A_2-A_3$, and not a hyperelliptic curve by Lemma 1.2(1).

REMARK. M in Example 2.7 does not satisfy the condition of Lemma 1.2(2) for $p=3$. But M has unique g_d^1 , because M has a canonical divisor $(dx/y)=2A_1+2A_2+2A_3$ and by [4] (III.8.7).

§ 3.

In the following sections we give some applications of our results in § 2. At first we will prove the following Theorem, which have been obtained by Namba [7] and improved by Kato [5] in the case $d=p$ a prime number.

THEOREM 3.1. *Let M and M' be defined by the following equations;*

$$y^d - (x-a_1)^{r_1} \cdots (x-a_s)^{r_s} = 0 \cdots \cdots \cdots \text{i)}$$

and

$$\tilde{y}^d - (\tilde{x}-b_1)^{t_1} \cdots (\tilde{x}-b_s)^{t_s} = 0 \cdots \cdots \cdots \text{ii)}$$

respectively, where $1 \leq r_i \leq d-1$, $1 \leq t_i \leq d-1$, $\sum r_i = \sum t_i \equiv 0 \pmod{d}$. Assume M satisfies the conditions in Theorem 2.4(2), and M and M' are birationally equivalent. Then, by changing the indices suitably, we have;

(1) *there exists $A \in \text{Aut}(\mathbf{P}_1)$ satisfying $b_i = Aa_i$ ($1 \leq i \leq s$), and*

$$\#) \begin{cases} \text{ord}_p t_i = \text{ord}_p r_i & \text{if } \text{ord}_p r_i < \text{ord}_p d \\ \text{ord}_p t_i \geq \text{ord}_p d & \text{if } \text{ord}_p r_i \geq \text{ord}_p d \end{cases} \quad (1 \leq i \leq s)$$

for each prime number p dividing d .

(2) *if $(r_1, d)=1$, then $r_1/t_1 \in (Z/dZ)^\times$ and $(r_1/t_1)t_i \equiv r_i \pmod{d}$ ($1 \leq i \leq s$).*

(3) *if d is square free, then $r_1 t_i \equiv t_1 r_i \pmod{d}$ ($2 \leq i \leq s$).*

PROOF. (1) The proof owes to the uniqueness of g_d^1 (Theorem 2.4(2)), and goes almost same way as in the proof of Theorem 1.1 in [6]. Let $\varphi: M \rightarrow M'$ be the birational map. As M has unique g_d^1 , there exists $A \in \text{Aut } \mathbf{P}_1$ satisfying a commutative diagram;

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & M' \\ x \downarrow & & \downarrow \tilde{x} \\ P_1 & \xrightarrow{A} & P_1. \end{array}$$

Thus we may assume $Aa_i = b_i$ for $i=1, \dots, s$. Let M'' be a curve defined by $z^d - (u - A^{-1}b_1)^{t_1} \dots (u - A^{-1}b_s)^{t_s} = 0$ and $\phi_A = \phi$ be a birational map from M' to M'' defined by $(\tilde{x}, \tilde{y}) \rightarrow (u, z) = (A^{-1}\tilde{x}, c\tilde{y}/(\tilde{x} - \gamma)^{k'})$, where c is a suitable constant, $\gamma = A(\infty)$ and $k' = (\sum t_i)/d$ ([6]). Put $w = z \cdot \phi \cdot \varphi$, which is a meromorphic function on M . Then M is also defined by

$$w^d - (x - a_1)^{t_1} \dots (x - a_s)^{t_s} = 0 \dots \dots \dots \text{i').}$$

As both i) and i') define the ramification type of the same cyclic covering $x: M \rightarrow P_1$, we can see #) by considering a covering map $M/\langle T^{p^{\alpha} r d} \rangle \rightarrow P_1$ induced by x .

(2), (3) Put $v = w^{r_1}/y^{t_1}$, then we have;

$$v^d - (x - a_2)^{r_1 t_2 - r_2 t_1} \dots (x - a_s)^{r_1 t_s - r_s t_1} = 0 \dots \dots \text{iii).}$$

Put $[C(M):C(x, v)] = t$. As $C(M) \supset C(x, v) \supset C(x)$ are cyclic extensions, $v^{d/t}$ is in $C(x)$ and $r_1 t_i - t_1 r_i \equiv 0 \pmod t$ ($2 \leq i \leq s$) by iii). Moreover we can see that s numbers $(r_1 t_i - t_1 r_i)/t$ ($2 \leq i \leq s$) and d/t have no common divisor and $G.C.D.(r_1, t_1, d) = (r_1, t_1, d)$ divides t . On the other hand $C(x, v)$ is the function field of the curve $M/\langle T^{d/t} \rangle$. Assume $d \neq t$, and take a prime number q dividing d/t . Then the curve $M/\langle T^q \rangle$ is defined by the following two equations simultaneously;

$$y^q - (x - a_1)^{r_1} \dots (x - a_s)^{r_s} = 0 \dots \dots \dots \text{A)}$$

and

$$v^q - (x - a_2)^{(r_1 t_2 - r_2 t_1)/t} \dots (x - a_s)^{(r_1 t_s - r_s t_1)/t} = 0 \dots \dots \text{B).}$$

Now we will show $r_1 \not\equiv 0 \pmod q$. In fact this is obvious when $(r_1, d) = 1$. Next we consider the case d is square free. From #) we have $(r_1, t_1, d) = (r_1, d)$. As d is square free and $(r_1, t_1, d) | t$, $(d/t, r_1, d) = (d/t, r_1) = 1$ and $(r_1, q) = 1$. Thus a_1 is a branch point of the covering $x: M/\langle T^q \rangle \rightarrow P_1$ by A). But this contradicts to B). So we have $t = d$ and

$$r_1 t_i - t_1 r_i \equiv 0 \pmod d \quad (2 \leq i \leq s).$$

When $(r_1, d) = 1$, then $(t_1, d) = 1$ by #, and we get (2). \square

REMARK. Conversely if there exists $A \in \text{Aut}(P_1)$ as in (1) and we have $(r_1/t_1)t_i \equiv r_i \pmod d$ ($2 \leq i \leq s$), then M and M' are birationally equivalent ([6]).

§ 4.

Next we consider a covering map $\pi': M' \rightarrow M$ with a cyclic d -gonal curve M defined by $*$) of genus g and a d' -gonal curve M' of genus g' .

THEOREM 4.1. Assume $d=d'$. Then;

- (1) M' is also a cyclic d -gonal curve.
- (2) If M satisfies the conditions of Theorem 2.4(2) and π' is normal, then the Galois group of π' is cyclic, dihedral, tetrahedral, octahedral or icosahedral.

PROOF. (1) Easily from Lemma 1.3(1). (2) Let T (resp. T') be the automorphism of order d on M (resp. M') as in § 2. By the commutative diagram in Lemma 1.3 and the uniqueness of g_d^1 on M we may assume that T' induces T . For each prime number q dividing d , we have a commutative diagram;

$$\begin{array}{ccc} M' & \longrightarrow & M'/\langle T'^q \rangle \\ \downarrow & & \downarrow \\ M & \longrightarrow & M/\langle T^q \rangle. \end{array}$$

Let g'_q be genus of $M'/\langle T'^q \rangle$. As $g \leq g'$ and $g_q \leq g'_q$, M' is also satisfying the conditions in Theorem 2.4(2). Then M' has a unique g_d^1 . By Lemma 1.3(2) we have our results. \square

THEOREM 4.2. Assume $d \leq d'$. If d and d' satisfy the conditions of Proposition 2.3. on M , then d divides d' .

PROOF. Let D' be a positive divisor of degree d' on M' such that $|D'|$ has projective dimension 1. Assume $Nm_\pi D'$ has some common point with $Nm_\pi E$ for each $E \in |D'|$. Then each $E \in |D'|$ has some common point with $\pi^* Nm D'$. On the other hand if E and E' in $|D'|$ have common points, then $E=E'$ by the minimality of d' . Hence $|D'|$ should be a finite set. This is a contradiction. Thus there is a meromorphic function h of degree d' on M' and $Nm[h]$ is also of degree d' on M . By Proposition 2.3 we have $d|d'$. \square

COROLLARY 4.3. Let $\pi': M' \rightarrow M$ be an unramified covering of degree q with a cyclic p -gonal curve M of genus g , where p and q are distinct prime numbers. Assume $g > p^2q - 2p + 1$. Then;

- (a) M' is a pq -gonal curve with a unique g_{pq}^1 .
- (b) Let $\phi: M' \rightarrow \mathbf{P}_1^1$ be the covering map defined by g_{pq}^1 in a), then;
 - (b-i) ϕ is not cyclic (i.e., M' is not a cyclic pq -gonal curve).
 - (b-ii) if $p \nmid q-1$, then ϕ is not normal.

PROOF. (a) Let $h: M \rightarrow P_1$ be the covering map of degree p , then $h \circ \pi'$ is a meromorphic function of degree pq on M' . For $g > p^2q - 2p + 1 > (pq-1)(p-1)$, M' is pm -gonal ($1 \leq m \leq q-1$) or pq -gonal by Theorem 4.2. (see the remark of Proposition 2.3). Now we assume that M' is pq -gonal. Let ϕ be a meromorphic function of degree pq on M' . Put $K = C(\phi, h \circ \pi')$ and $[C(M'): K] = t$. As the genus g' of M' is $q(g-1)+1$, we have $g' > (pq-1)^2$ and $t \neq 1$. Consider the following diagram;

$$\begin{array}{ccc} C(M') & \supset & K \supset C(\phi) \\ \cup & & \cup \\ C(M) & \supset & C(h \circ \pi'). \end{array}$$

If $t=q$, then $[K: C(h \circ \pi')] = p$ and genus of $K = g$ ($\because \pi'$ is unramified and $(p, q)=1$). For $g > (p-1)^2$, $K = C(h \circ \pi')$. This is a contradiction. If $t=p$, then $K \supset C(h \circ \pi')$ is an unramified extension. As $C(h \circ \pi')$ is of genus 0, this is a contradiction. Thus we have $t=pq$ and M' has a unique g_{pq}^1 . If M' is pm -gonal ($1 \leq m \leq q-1$) and ϕ is a meromorphic function of degree pm on M' , then $[C(M'): C(\phi, h \circ \pi')] = p$ by $(p, q)=1$ and $g' > (pm-1)(pq-1)$. This is a contradiction.

(b-i) We may assume $h \circ \pi' = \phi$ by (a). If ϕ is cyclic, then there exists an automorphism T' on M' of order p , and we have a commutative diagram;

$$\begin{array}{ccc} M' & \longrightarrow & M'/\langle T' \rangle \\ \pi' \downarrow & & \downarrow \pi \\ M & \xrightarrow{h} & M/\langle T \rangle = P_1, \text{ where } \pi' \text{ is unramified.} \end{array}$$

For $(p, q)=1$, π is unramified. This is a contradiction. (b-ii) Assume ϕ is normal with galois group G . If $p < q$ and $p \nmid q-1$, it is well known that G is cyclic. But this can not be happened by (a). If $p > q$, then G has a unique normal subgroup $\langle T' \rangle$ of index q generated by T' . Thus we have a same commutative diagram as in the proof of (b-i). This is also a contradiction. \square

§ 5.

We consider a covering $\pi': M \rightarrow N$, where M is cyclic d -gonal and N is e -gonal. Put $\deg \pi = n$ and $d' = ne$.

THEOREM 5.1. Assume d and d' satisfy the conditions of Proposition 2.3. Then e divides d . Moreover if $u: M \rightarrow M/\langle T^{d/e} \rangle$ is the canonical map, then there exists a covering map $v: M/\langle T^{d/e} \rangle \rightarrow N$ satisfying $\pi' = v \circ u$. Especially when $d = d' = ne$, N is isomorphic to $M/\langle T^{d/e} \rangle$.

PROOF. Let $\phi_N: N \rightarrow \tilde{P}_1$ be the covering over Riemann sphere \tilde{P}_1 of degree e . Then $\phi_N \circ \pi'$ is a meromorphic function on M of degree $d' = ne$. By Proposition 2.3, d divides $ne = d'$, and we have a commutative diagram;

$$\begin{array}{ccc} M' & \xrightarrow{h} & P_1 = M/\langle T \rangle \\ \pi' \downarrow & & \downarrow \tilde{\pi} \\ N & \xrightarrow{\phi_N} & \tilde{P}_1, \end{array}$$

with a rational function $\tilde{\pi}$ of degree d'/d and the canonical map h . The function fields $C(N)$ and $C(P_1)$ are linearly independent over $C(\tilde{P}_1)$ for the minimality of e . Then there exists a e -gonal curve \tilde{M} with a function field $C(\tilde{M})$ isomorphic to $C(P_1) \otimes_{C(\tilde{P}_1)} C(N)$. By the universal property of $C(\tilde{M})$ we have the following commutative diagram;

$$\begin{array}{ccccc} M & & h & & \\ & \searrow & & \searrow & \\ & \tilde{M} & \xrightarrow{\tilde{\phi}} & P_1 = M/\langle T \rangle & \\ \pi \swarrow & & & \downarrow \tilde{\pi} & \\ & N & \xrightarrow{\phi_N} & \tilde{P}_1 & \end{array}$$

where $\deg \tilde{\phi} = e$ and $\deg \tilde{\pi} = ne/d$. We can see that e divides d . As h is a cyclic extension, $\tilde{M} \cong M/\langle T^{d/e} \rangle$. \square

EXAMPLE 5.2. Let M be the cyclic pq -gonal curve defined in Example 2.6 with $p \geq q$, $s \geq 2pq + 1$ and $(p-1)^2 < (q-1)(s-2)/2$. Then any covering $\pi: M \rightarrow N$ of degree p (resp. q) with a q (resp. p)-gonal curve N is birational to the cyclic q (resp. p)-gonal curve defined by $y^q - (x-a_1)^{r_1} \cdots (x-a_s)^{r_s} = 0$ (resp. $y^p - (x-a_1)^{r_1} \cdots (x-a_s)^{r_s} = 0$).

§6.

Let M be a cyclic d -gonal curve with a unique g_d^1 defined by

$$y^d - (x-a_1)^{r_1} \cdots (x-a_s)^{r_s} = 0, \quad \sum r_i \equiv 0 \pmod{d}, \dots \dots *$$

$(r_i, d) = 1$ for all i , here we can take ∞ as one of a_i 's.

Let T be the automorphism of order d as in §2, and $\phi: M \rightarrow M/\langle T \rangle$ be the canonical map. We will determine the equation $*$, which defines M having an automorphism $V (\notin \langle T \rangle)$ of order N .

For the uniqueness of g_d^1 , we have $V\langle T \rangle V^{-1} = \langle T \rangle$ and V induces an automorphism \tilde{V} on $M/\langle T \rangle = P_1(x)$. Let $C(x)$ and $C(u)$ be the function fields of $M/\langle T \rangle$ and $M/\langle V, T \rangle$ respectively. Then $\pi': M/\langle T \rangle \rightarrow M/\langle T, V \rangle$ is a cyclic covering of order N' ($N' | N$) and we may assume $\pi'^*u = x^{N'}$.

Before considering generally, we study the following two cases;

$$\text{Case 1) } \langle T \rangle \cap \langle V \rangle = \langle T \rangle, \quad \text{Case 2) } \langle T \rangle \cap \langle V \rangle = \{1\}.$$

Case 1) $\langle T \rangle \cap \langle V \rangle = \langle T \rangle$

We can see that $d | N$ and $N' = N/d$. We may assume $V^{N/d} = T$ and $\tilde{V}^*x = \zeta'x$ with a primitive N' -th root ζ' of 1. We denote the set {fixed point of \tilde{V} } by $F(\tilde{V})$. Then $\#F(\tilde{V}) = 2$.

$$\text{Case 1-a) } \#F(\tilde{V}) \cap \{a_1, \dots, a_s\} = 2$$

We may assume that two elements of the above set are $a_{s-1} = 0$ and $a_s = \infty$. As \tilde{V} acts on $\{a_1, \dots, a_{s-2}\}$ faithfully, M can be defined by;

$$\begin{aligned} \text{A) } \quad y^d &= x \left\{ \prod_{t=1}^k \prod_{j=1}^{N/d} (x - \zeta'^{j-1} c_t)^{m_{N/d \cdot (t-1) + j}} \right\}, \\ 1 + \sum_{t=1}^k \sum_{j=1}^{N/d} m_{N/d \cdot (t-1) + j} &\not\equiv 0 \pmod{d}, \end{aligned}$$

where $(m_*, d) = 1$, and c_t ($\neq 0$) are distinct complex numbers satisfying

$$\{\zeta'^{j-1} c_t | 1 \leq j \leq N/d\} \cap \{\zeta'^{j-1} c_s | 1 \leq j \leq N/d\} = \emptyset \quad \text{for } t \neq s.$$

By acting V^* on both sides of A), we have;

$$\text{B) } \quad (T^*y)^d = \zeta'^M \left\{ \prod_{t=1}^k \prod_{j=1}^{N/d} (x - \zeta'^{j-2} c_t)^{m_{N/d \cdot (t-1) + j}} \right\} x,$$

$$\text{where } M = 1 + \sum_{t=1}^k \sum_{j=1}^{N/d} m_{N/d \cdot (t-1) + j}.$$

By the proof of Theorem 3.1 and comparing A) with B), there exists a positive integer v ($1 \leq v < d$, $(v, d) = 1$) satisfying $v \cdot m_{N/d \cdot (t-1) + j} \equiv m_{N/d \cdot (t-1) + j+1} \pmod{d}$ ($1 \leq j \leq N/d - 1$), and $vm_{N/d \cdot t} \equiv m_{N/d \cdot (t-1) + 1} \pmod{d}$. But in this case, $v \cdot 1 \equiv 1 \pmod{d}$.

Thus we have $v = 1$ and $m_{N/d \cdot (t-1) + 1} = \dots = m_{N/d \cdot t} \stackrel{\text{put}}{=} r_t$ ($t = 1 \leq t \leq k$). The equation A) is;

$$\text{I) } \quad y^d = x \left\{ \prod_{t=1}^k \prod_{j=1}^{N/d} (x - \zeta'^{j-1} c_t)^{r_t} \right\} = x \cdot \prod_{t=1}^k (x^{N/d} - b_t)^{r_t},$$

As $V^*y^d = \zeta'y^d$ and V is of order N , we have $V^*y = \eta y$, where η satisfies $\eta^d = \zeta'$ and $\eta^{N'}$ is a primitive N/N' ($=d$)-th root of 1.

PROPOSITION 6.1a). Case 1-a happens if and only if M is defined by 1) with $d|N$, $(r_t, d)=1$ ($t=1, \dots, k$) and $N/d \sum_{t=1}^k r_t + 1 \not\equiv 0 \pmod{d}$. V is defined by

$$V^*x = \zeta'x \quad \text{and} \quad V^*y = \eta y, \dots \dots \dots 1)$$

where ζ' is a primitive N' -th root of 1, η satisfies $\eta^d = \zeta'$ and $\eta^{N'}$ is a primitive d -th root of 1 (for example, $\eta = e^{2\pi i/N}$ and $\zeta' = e^{2\pi i/N'}$ satisfy these conditions).

Case 1-b) $\#F(\tilde{V}) \cap \{a_1, \dots, a_s\} = 1$

We may assume that the element of the above set is a_s . There exists a point $P \in M$ such that $\phi(P) \notin \{a_1, \dots, a_s\}$ and $V(P) \in \langle T \rangle P = \langle V^{N/d} \rangle P$. Then $V^d(P) = P$. If $(d, N/d) = r \neq 1$, then $T^{d/r}P = V^{N/d \cdot d/r}P = P$. This contradicts to $\phi(P) \notin \{a_1, \dots, a_s\}$. Thus $(d, N/d) = 1$ and $\langle V^d \rangle \cap \langle V^{N/d} \rangle = \{1\}$. We have $C(M) = C(M/\langle V^{N/d} \rangle) \otimes_{C(M/\langle V \rangle)} C(M/V^d)$. Assume $\phi(P) = \infty$, $a_s = 0$ and $\pi'^*u = x^{N/d}$. As $M/\langle V^d \rangle \rightarrow M/\langle V \rangle = P_1(u)$ is cyclic of degree d , $C(M/\langle V^d \rangle)$ is defined by $y^d = u \prod_{t=1}^k (u - b_t)^{n_t}$, with $(n_t, d) = 1$ ($t=1, \dots, e$) and $1 + n_1 + \dots + n_k \not\equiv 0 \pmod{d}$. Then M is defined by $y^d = x^{N/d} (x^{N/d} - b_1)^{n_1} \dots (x^{N/d} - b_k)^{n_k}$. For $(d, N/d) = 1$, M can be defined by the following equation;

$$\text{II)} \quad y^d = x \cdot (x^{N/d} - b_1)^{r_1} \dots (x^{N/d} - b_k)^{r_k}, \quad \text{with} \quad 1 + \sum r_t \not\equiv 0 \pmod{d}.$$

After all, we have;

PROPOSITION 6.1b). Case 1-b) happens if and only if $(N/d, d) = 1$ and M is defined by II) with $(r_t, d) = 1$ and $1 + \sum_{t=1}^e r_t \not\equiv 0 \pmod{d}$. V is defined by;

$$V^*x = \zeta'x \quad \text{and} \quad V^*y = \eta y, \dots \dots \dots 2)$$

where ζ' is a primitive N' -th root of 1, η satisfies $\eta^d = \zeta'$ and $\eta^{N'}$ is a primitive d -th root of 1.

Case 1-c) $\#F(\tilde{V}) \cap \{a_1, \dots, a_s\} = \emptyset$

By the same way as in Case 1-b), we have;

PROPOSITION 6.1c). Case 1-c) happens if and only if $(N/d, d) = 1$ and M is defined by;

$$\text{III)} \quad y^d = (x^{N/d} - b_1)^{r_1} \dots (x^{N/d} - b_k)^{r_k}$$

with $(r_t, d) = 1$ and $\sum_{t=1}^k r_t \equiv 0 \pmod{d}$. V is defined by;

$$V^*x = \zeta'x \quad \text{and} \quad V^*y = \zeta''y, \quad \dots\dots\dots 3)$$

where ζ' (resp. ζ'') is a primitive N' (resp. d)-th root of 1.

Case 2) $\langle T \rangle \cap \langle V \rangle = \{1\}$

The automorphism \tilde{V} on $M/\langle T \rangle$ induced by V is of order N , and we may assume that $\tilde{V}^*x = \zeta x$ with a primitive N -th root ζ of 1.

Case 2-a) $\#\{a_1, \dots, a_s\} \cap F(\tilde{V}) = 2$ and

Case 2-b) $\#\{a_1, \dots, a_s\} \cap F(\tilde{V}) = 1$

By the same way as in Case 1-a), M can be defined by

$$\text{IV)} \quad y^d = x \prod_{t=1}^k (x^N - b_t)^{r_t}, \quad \text{with } (r_t, N) = 1.$$

In Case 2-a) (resp. 2-b), $N \sum_{t=1}^k r_t + 1 \not\equiv 0$ (resp. $\equiv 0$) mod d . As V satisfies $V^*y^d = \zeta \cdot y^d$ and V is of order N , V is defined by;

$$V^*x = \zeta x \quad \text{and} \quad V^*y = \xi \cdot y, \quad \dots\dots\dots 4)$$

where ξ is a N -th root of 1 satisfying $\xi^d = \zeta$. $\therefore (d, N) = 1$ and ξ is also a primitive N -th root of 1. After all we have;

PROPOSITION 6.2. Case 2-a) (resp. 2-b)) happens if and only if $(N, d) = 1$ and M is birational to the curve defined by IV) with $(r_t, N) = 1$ and $N \sum_{t=1}^k r_t + 1 \not\equiv 0$ (resp. $\equiv 0$) mod d . V is defined by 4) with a primitive N -th root ξ of 1 and $\zeta = \xi^d$.

Case 2-c) $\#\{a_1, \dots, a_s\} \cap F(\tilde{V}) = 0$

By the same way as in Case 1-a), M is birational to the curve defined by

$$y^d = \left\{ \prod_{t=1}^k \prod_{j=1}^N (x - \zeta^{j-1} b_t)^{m_{N(t-1)+j}} \right\} \quad \text{with} \quad \sum_{t=1}^k \sum_{j=1}^N m_{N/d \cdot (t-1)+j} \equiv 0 \pmod{d}$$

and $(m_*, d) = 1$. Moreover there exists a positive integer v ($1 \leq v \leq d-1$, $(v, d) = 1$) satisfying $vm_{N(t-1)+j} \equiv m_{N(t-1)+j+1} \pmod{d}$ ($1 \leq j \leq N-1$), and $vm_{N \cdot t} \equiv m_{N(t-1)+1} \pmod{d}$. We see $v^N \equiv 1 \pmod{d}$. Thus M is defined by

$$\text{V)} \quad y^d = \prod_{t=1}^k \prod_{j=1}^N (x - \zeta^{j-1} b_t)^{n_t v^{j-1}}$$

with positive integers n_t satisfying $\sum_{t=1}^k \sum_{j=1}^N n_t v^{j-1} \equiv 0 \pmod{d}$ and $(n_*, d) = 1$. Put $R = \sum n_t$ and $S = \sum v^{j-1}$. Then $RS \equiv 0 \pmod{d}$. By acting V^* on the both sides of V again, we have

$$\begin{aligned}
(V^*y)^d &= \prod_{t=1}^k \prod_{j=1}^N (\zeta x - \zeta^{j-1} b_t)^{n_t v^{j-1}} \\
&= \zeta^{RS} \prod_{t=1}^k \prod_{j=1}^N (x - \zeta^{j-2} b_t)^{n_t v^{j-1}} \\
&= \begin{cases} \zeta^{RS} y^{vd} / \prod_{t=1}^k (x - \zeta^{N-1} b_t)^{n_t (v^{N-1})}, & \zeta^{RS} \neq 1 \quad (\text{if } RS \not\equiv 0 \pmod{N}). \\ \text{or} \\ y^{vd} / \prod_{t=1}^k (x - \zeta^{N-1} b_t)^{n_t (v^{N-1})} & (\text{if } RS \equiv 0 \pmod{N}). \end{cases}
\end{aligned}$$

Then we have;

$$V^*y = \begin{cases} \eta \zeta^{RS/d} y^v / \prod_{t=1}^k (x - \zeta^{N-1} b_t)^{n_t (v^{N-1})/d}, & (\text{if } RS \not\equiv 0 \pmod{N}) \cdots \text{V-i)} \\ \text{or} \\ \eta y^v / \prod_{t=1}^k (x - \zeta^{N-1} b_t)^{n_t (v^{N-1})/d}, & (\text{if } RS \equiv 0 \pmod{N}) \cdots \text{V-ii).} \end{cases}$$

where η is some d -th root (not necessarily primitive) of 1.

Assume $RS \not\equiv 0 \pmod{N}$. Using V-i) repeatedly, we have;

$$\begin{aligned}
V^*N y &= \eta^S \zeta^{(RS/d)S} y^{v^N} / \left[\left\{ \prod_{t=0}^{N-1} \prod_{l=1}^k (\zeta^l x - \zeta^{N-1} b_t)^{n_t} \right\}^{v^{N-1-l}} \right]^{(v^{N-1})/d} \\
&= \eta^S \zeta^{(RS/d)S} y^{v^N} / \zeta^{R(v^{N-2} + 2v^{N-3} + \cdots + (N-1)v^0)} \left[\left\{ \prod_{t=0}^{N-1} \prod_{l=1}^k (x - \zeta^{N-1-l} b_t)^{n_t} \right\}^{v^{N-1-l}} \right]^{(v^{N-1})/d} \\
&= \eta^S \zeta^{(RS/d)S - R(S^2 - NS)/d} y^{v^N} / (y^d)^{(v^{N-1})/d} = \eta^S \zeta^{RNS/d} y = \eta^S y \quad (\because RS \equiv 0 \pmod{d}).
\end{aligned}$$

For $V^*N y = y$, $\eta^S = 1$ should be held.

When $RS \equiv 0 \pmod{N}$, by the same way as above, we have;

$$V^*N y = \eta^S \zeta^{-R(S^2 - NS)/d} y^{v^N} (y^d)^{(v^{N-1})/d} = \eta^S \zeta^{-RS^2/d} y.$$

Thus η should satisfy $\eta^S = \zeta^{RS^2/d}$.

PROPOSITION 6.3. *Case 2-c) happens if and only if M is birational to the curve defined by V) with $v^N \equiv 1 \pmod{d}$ and $RS \equiv 0 \pmod{d}$. If $RS \not\equiv 0$ (resp. $RS \equiv 0$) \pmod{N} , V is defined by $V^*x = \zeta x$ and V-i) (resp. V-ii) with d -th root η of 1 satisfying $\eta^S = 1$ (resp. $\eta^S = \zeta^{RS^2/d}$), here η is not necessarily primitive (for example, $\eta = 1$ (resp. $\eta = \zeta^{RS^2/d}$) satisfies $\eta^S = 1$ (resp. $\eta^S = \zeta^{RS^2/d}$)).*

General case $\langle T \rangle \cap \langle V' \rangle = \langle V^{N'} \rangle = \langle T^{d'} \rangle$.

We can obtain the equations of M and V as follows. We may assume that $N' | N$ and $d' | d$, then $d/d' = N/N'$. The case $d' = 1$ is exactly same as the case 1) (Propositions 6-1a~c)).

When $d' > 1$, put $M' = M / \langle T \rangle \cap \langle V \rangle$. Then M' is d' -gonal with a unique $g_{d'}^1$ having an automorphism V' ($= V \bmod \langle V^{d'} \rangle$) of order d' . We can apply Proposition 6.2 or 6.3, and M' is defined by an equation of type IV) or V).

For example, assume M' is defined by;

$$y'^{d'} = \prod_{t=1}^{k'} \prod_{j=1}^{N'} (x - \zeta'^{j-1} b'_t)^{n'_t v'^{j-1}} \quad (\text{cf. V})$$

with $(n'_*, d') = (v', d') = 1$, $1 \leq v' \leq d' - 1$, and $R'S' \equiv 0 \pmod{d'}$, where $R' = \sum_{t=1}^{k'} n'_t$, $S' = \sum_{j=1}^{N'} v'^{j-1}$ and a primitive N' -th root ζ' of 1. Moreover, assume $R'S' \not\equiv 0 \pmod{N'}$. Then V' is defined by;

$$\begin{cases} V'^* x = \zeta' x \\ V'^* y' = \eta' \zeta'^{R'S'/d'} y'^{v'} / \prod_{t=1}^{k'} (x - \zeta'^{N'-1} b'_t)^{n'_t (v'^{N'-1})/d'} \end{cases} \quad (\text{cf. V-i}),$$

with d' -th root η' (not necessarily primitive) of 1 satisfying $\eta'^{s'} = 1$. Put $y' = y^{d/d'}$, we can have the equation of M ;

$$y^d = \prod_{t=1}^{k'} \prod_{j=1}^{N'} (x - \zeta'^{j-1} b'_t)^{n'_t v'^{j-1}}. \dots\dots\dots \text{VI})$$

As M is defined by *), we have $R'S' \equiv 0 \pmod{d}$, $(n'_*, d) = (v', d) = 1$ and $v'^N \equiv 1 \pmod{d}$. Thus V on M is defined by;

$$\begin{cases} V^* x = \zeta' x \\ V^* y = \eta \zeta'^{R'S'/d} y^{v'} / \prod_{t=1}^{k'} (x - \zeta'^{N'-1} b'_t)^{n'_t (v'^{N'-1})/d}, \end{cases}$$

where η satisfies $\eta^{d/d'} = \eta'$. We can see $V^{*N'} y = \eta^{S'} y$. As V is of order N , $\eta^{S'}$ should be a primitive N/N' ($= d/d'$) root of 1. When $(S', d/d') = 1$, $\eta' = 1$, and $\eta = \exp(2\pi i d'/d)$ satisfies these conditions,

Considering the other cases, we finally have;

THEOREM 6.4. *Let M be a cyclic d -gonal curve with a unique g_d^1 defined by *) with an automorphism V ($\notin \langle T \rangle$) of order N . Then M and V are determined as the following types;*

I) *Let d' (> 1) and N' (> 1) be two integers satisfying $d' | d$, $N' | N$ and $d/d' = N/N' \neq 1$.*

I-i) *M is a curve defined by the equation*

$$y^d = \prod_{t=1}^{k'} \prod_{j=1}^{N'} (x - \zeta'^{j-1} b'_t)^{n'_t v'^{j-1}} \dots\dots\dots \text{VI})$$

with $1 \leq v' \leq d' - 1$, $(n'_, d) = (v', d) = 1$ and $S'R' \equiv 0 \pmod{d}$.*

If $S'R' \not\equiv 0 \pmod{N'}$, then V is defined by

$$\begin{cases} V^*x = \zeta'x \\ V^*y = \eta \zeta'^{R'S'/d} y^{v'} / \prod_{t=1}^{k'} (x - \zeta'^{N'-1} b'_t)^{n'_t (v'^{N'-1})/d}, \end{cases}$$

where η is a d -th root (not necessarily primitive) of 1 such that $\eta^{S'}$ is a primitive d/d' -th root of 1. (for example, when $(S', d/d')=1$, $e^{2\pi i d'/d}$ can be taken as η).

If $S'R' \equiv 0 \pmod{N'}$, V is defined by

$$\begin{cases} V^*x = \zeta'x \\ V^*y = \eta y^{v'} / \prod_{t=1}^{k'} (x - \zeta'^{N'-1} b'_t)^{n'_t (v'^{N'-1})/d}, \end{cases}$$

where η is a d -th root (not necessarily primitive) of 1 such that $\eta \zeta'^{-R'S'^2/d}$ is a primitive d/d' -th root of 1. (for example, when $(S', d/d')=1$, we can take $\zeta'^{R'S'/d} \zeta_{d/d'}$ as η , where $\zeta_{d/d'}$ is a primitive d/d' -th root of 1). (cf. Prop. 6.3)

I-ii) If $(d', N')=1$, we have an additional type;

$$y^d = x \prod_{t=1}^k (x^{N'} - b_t)^{r_t}$$

with $(r_t, N)=1$. In this case V is defined by;

$$V^*y = \xi y \quad \text{and} \quad V^*x = \xi^d x,$$

where ξ is a primitive N -th root of 1. (cf. Prop. 6.2)

II) In case of $d|N$, in addition to 1), we have other types of M and V as follows;

II-i) M and V in Proposition 6.1a).

II-ii) In addition to II-i), M and V in Proposition 6.1b) and 6.1c), provided $((d, N/d)=1$.

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